$$P(Y \geq k) = \frac{1}{1+e^{a(\delta_k-\theta)}}$$

$$P(Y=0) = 1 - P(Y \geq 1)$$
$$P(Y=1) = P(Y \geq 1) - P(Y \geq 2)$$
$$P(Y=2) = P(Y \geq 2) - P(Y \geq 3)$$
$$\vdots \qquad \vdots$$
$$P(Y=k) = P(Y \geq k)$$

Intersection Cat 0 and Cat 1

$$P(Y=0) = P(Y=1)$$

$$1 - P(Y \geq 1) = P(Y \geq 1) - P(Y \geq 2)$$

$$1 = 2\,P(Y \geq 1) - P(Y \geq 2)$$

$$1 = 2\,\frac{1}{1+e^{a(\delta_1-\theta)}} - \frac{1}{1+e^{a(\delta_2-\theta)}}$$

$$1 = \frac{2 + 2e^{a(\delta_2-\theta)} - 1 - e^{a(\delta_1-\theta)}}{\left[1+e^{a(\delta_1-\theta)}\right]\left[1+e^{a(\delta_2-\theta)}\right]}$$

$$\left[1+e^{a(\delta_1-\theta)}\right]\left[1+e^{a(\delta_2-\theta)}\right] = 1 + 2e^{a(\delta_2-\theta)} - e^{a(\delta_1-\theta)}$$

$$\cancel{1} + \underbrace{e^{a(\delta_2-\theta)}} + \underbrace{e^{a(\delta_1-\theta)}} + e^{a(\delta_1-\theta)+a(\delta_2-\theta)} = \cancel{1} + 2\underbrace{e^{a(\delta_2-\theta)}} - \underbrace{e^{a(\delta_1-\theta)}}$$

$$e^{a\delta_1 - a\theta + a\delta_2 - a\theta} = e^{a\delta_2 - a\theta} - 2e^{a\delta_1 - a\theta}$$

$$\frac{e^{a\delta_1 + a\delta_2 - a\theta}}{\cancel{e^{a\theta}}} = \frac{e^{a\delta_2}}{\cancel{e^{a\theta}}} - \frac{2e^{a\delta_1}}{\cancel{e^{a\theta}}}$$

$$e^{a\delta_1 + a\delta_2 - a\theta} = e^{a\delta_2} - 2e^{a\delta_1}$$

$$\frac{e^{a\delta_1 + a\delta_2}}{e^{a\theta}} = e^{a\delta_2} - 2e^{a\delta_1}$$

$$e^{a\theta} = \frac{e^{a\delta_1 + a\delta_2}}{e^{a\delta_2} - 2e^{a\delta_1}}$$

$$\theta = \frac{\ln\left[\frac{e^{a\delta_1 + a\delta_2}}{e^{a\delta_2} - 2e^{a\delta_1}}\right]}{a} \quad //$$

page 1

Intersection Cat 1 and Cat 2

$$P(Y=1) = P(Y=2)$$

$$P(Y \geqslant 1) - P(Y \geqslant 2) = P(Y \geqslant 2) - P(Y \geqslant 3)$$

$$P(Y \geqslant 1) = 2P(Y \geqslant 2) - P(Y \geqslant 3)$$

$$\frac{1}{1+e^{a(\delta_1-\theta)}} = \frac{2}{1+e^{a(\delta_2-\theta)}} - \frac{1}{1+e^{a(\delta_3-\theta)}}$$

$$\frac{1}{1+e^{a(\delta_1-\theta)}} = \frac{2+2e^{a(\delta_3-\theta)} - 1 - e^{a(\delta_2-\theta)}}{\left[1+e^{a(\delta_2-\theta)}\right]\left[1+e^{a(\delta_3-\theta)}\right]}$$

$$\frac{1}{1+e^{a(\delta_1-\theta)}} = \frac{1+2e^{a(\delta_3-\theta)} - e^{a(\delta_2-\theta)}}{1+e^{a(\delta_3-\theta)} + e^{a(\delta_2-\theta)} + e^{a(\delta_2-\theta)+a(\delta_3-\theta)}}$$

$$\cancel{1+e^{a(\delta_3-\theta)}} + e^{a(\delta_2-\theta)} + e^{a(\delta_2-\theta)+a(\delta_3-\theta)} = \cancel{1}+\cancel{2}e^{a(\delta_3-\theta)} - e^{a(\delta_2-\theta)} + e^{a(\delta_1-\theta)} + 2e^{a(\delta_3-\theta)+a(\delta_1-\theta)} - e^{a(\delta_1-\theta)+a(\delta_2-\theta)}$$

$$\frac{e^{a\delta_2}}{\cancel{e^{a\theta}}} + \frac{e^{a\delta_2+a\delta_3-a\theta}}{\cancel{e^{a\theta}}} = \frac{e^{a\delta_3}}{\cancel{e^{a\theta}}} - \frac{e^{a\delta_2}}{\cancel{e^{a\theta}}} + \frac{e^{a\delta_1}}{\cancel{a\theta}} + 2\frac{e^{a\delta_3+a\delta_1-a\theta}}{\cancel{e^{a\theta}}} - \frac{e^{a\delta_1+a\delta_2-a\theta}}{\cancel{e^{a\theta}}}$$

$$e^{a\delta_2} + \frac{e^{a\delta_2+a\delta_3}}{e^{a\theta}} = e^{a\delta_3} - e^{a\delta_2} + e^{a\delta_1} + 2\frac{e^{a\delta_3+a\delta_1}}{e^{a\theta}} - \frac{e^{a\delta_1+a\delta_2}}{e^{a\theta}}$$

$$\frac{e^{a\delta_2+a\delta_3}}{e^{a\theta}} + \frac{e^{a\delta_1+a\delta_2}}{e^{a\theta}} - 2\frac{e^{a\delta_3+a\delta_1}}{e^{a\theta}} = e^{a\delta_3} + e^{a\delta_1} - 2e^{a\delta_2}$$

$$\frac{e^{a\delta_2+a\delta_3} + e^{a\delta_1+a\delta_2} - 2e^{a\delta_3+a\delta_1}}{e^{a\delta_3} + e^{a\delta_1} - 2e^{a\delta_2}} = e^{a\theta}$$

$$\boxed{\theta = \frac{\ln\left[\dfrac{e^{a\delta_1+a\delta_2} + e^{a\delta_2+a\delta_3} - 2e^{a\delta_1+a\delta_3}}{e^{a\delta_1} + e^{a\delta_3} - 2e^{a\delta_2}}\right]}{a}}$$

Intersection Cat 2 and Cat 3

$$P(Y=2) = P(Y=3)$$

$$P(Y\geq 2) - P(Y\geq 3) = P(Y\geq 3) - P(Y\geq 4)$$

$$P(Y\geq 2) = 2P(Y\geq 3) - P(Y\geq 4)$$

$$\frac{1}{1+e^{a(\delta_2-\theta)}} = \frac{2}{1+e^{a(\delta_3-\theta)}} - \frac{1}{1+e^{a(\delta_4-\theta)}}$$

$$\frac{1}{1+e^{a(\delta_2-\theta)}} = \frac{2+2e^{a(\delta_4-\theta)} - 1 - e^{a(\delta_3-\theta)}}{[1+e^{a(\delta_3-\theta)}][1+e^{a(\delta_4-\theta)}]}$$

$$\frac{1}{1+e^{a(\delta_2-\theta)}} = \frac{1+2e^{a(\delta_4-\theta)} - e^{a(\delta_3-\theta)}}{1+e^{a(\delta_4-\theta)}+e^{a(\delta_3-\theta)}+e^{a(\delta_3-\theta)+a(\delta_4-\theta)}}$$

$$\cancel{1}+\cancel{e^{a(\delta_4-\theta)}}+e^{a(\delta_3-\theta)}+e^{a(\delta_3-\theta)+a(\delta_4-\theta)} = \cancel{1}+\cancel{2}e^{a(\delta_4-\theta)} - e^{a(\delta_3-\theta)} + e^{a(\delta_2-\theta)} + 2e^{a(\delta_4-\theta)+a(\delta_2-\theta)} - e^{a(\delta_3-\theta)+a(\delta_2-\theta)}$$

$$\frac{e^{a\delta_3+a\delta_4-a\theta}}{\cancel{e^{a\theta}}} = \frac{e^{a\delta_4}}{\cancel{e^{a\theta}}} - \frac{2e^{a\delta_3}}{\cancel{e^{a\theta}}} + \frac{e^{a\delta_2}}{\cancel{e^{a\theta}}} + 2\frac{e^{a\delta_4+a\delta_2-a\theta}}{\cancel{e^{a\theta}}} - \frac{e^{a\delta_3+a\delta_2-a\theta}}{\cancel{e^{a\theta}}}$$

$$\frac{e^{a\delta_3+a\delta_4}}{e^{a\theta}} = e^{a\delta_4}+e^{a\delta_2}-2e^{a\delta_3}+2\frac{e^{a\delta_4+a\delta_2}}{e^{a\theta}} - \frac{e^{a\delta_3+a\delta_2}}{e^{a\theta}}$$

$$\frac{e^{a\delta_3+a\delta_4}+e^{a\delta_3+a\delta_2}-2e^{a\delta_4+a\delta_2}}{e^{a\theta}} = e^{a\delta_4}+e^{a\delta_2}-2e^{a\delta_3}$$

$$\theta = \frac{\ln\left[\dfrac{e^{a\delta_2+a\delta_3}+e^{a\delta_3+a\delta_4}-2e^{a\delta_2+a\delta_4}}{e^{a\delta_2}+e^{a\delta_4}-2e^{a\delta_3}}\right]}{a}$$

Intersection between Cat $\underline{\underline{j}}$ and Cat $\underline{\underline{j+1}}$

$$\theta = \frac{\ln\left[\dfrac{e^{a\delta_j + a\delta_{j+1}} + e^{a\delta_{j+1} + a\delta_{j+2}} - 2e^{a\delta_j + a\delta_{j+2}}}{e^{a\delta_j} + e^{a\delta_{j+2}} - 2e^{a\delta_{j+1}}}\right]}{a}$$

Special case when $a=1$

$$\theta = \ln\left[\frac{e^{\delta_j + \delta_{j+1}} + e^{\delta_{j+1} + \delta_{j+2}} - 2e^{\delta_j + \delta_{j+2}}}{e^{\delta_j} + e^{\delta_{j+2}} - 2e^{\delta_{j+1}}}\right]$$

Intersection between Cat k-1 and Cat k ↳ highest category.

$$P(Y=(k-1)) = P(Y=k)$$

$$P(Y \geqslant k-1) - P(Y \geqslant k) = P(Y \geqslant k)$$

$$P(Y \geqslant k-1) = 2P(Y \geqslant k)$$

$$\frac{1}{1+e^{a(\delta_{k-1}-\theta)}} = \frac{2}{1+e^{a(\delta_k-\theta)}}$$

$$1+e^{a(\delta_k-\theta)} = 2+2e^{a(\delta_{k-1}-\theta)}$$

$$\frac{e^{a\delta_k}}{e^{a\theta}} = 1+2\frac{e^{a\delta_{k-1}}}{e^{a\theta}}$$

$$e^{a\delta_k} = e^{a\theta}+2e^{a\delta_{k-1}}$$

$$e^{a\delta_k}-2e^{a\delta_{k-1}} = e^{a\theta}$$

$$\boxed{\theta = \frac{\ln\left[e^{a\delta_k}-2e^{a\delta_{k-1}}\right]}{a}}$$